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LETTER TO THE EDITOR

Extended coherent states and modified perturbation theory

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Abstract. An extended coherent state (ECS) for describing a system of two interacting quantum objects is considered. A modified perturbation theory based on using the ECSs is formulated.

1. Extended coherent states

Coherent states were constructed first by Schrödinger [1] and in the last 40 years of the 20th century were widely used in different problems of quantum physics [2]. There are many modifications of coherent states. Recall, for example, the spin coherent states introduced in [3, 4]. A general algebraic approach in the coherent state theory was developed in [4]. The coherent states for a particle on a sphere were applied in [5] to describe the rotator time evolution. Here we propose one more generalization of the theory by introducing the extended coherent state (ECS).

Consider a system of an oscillator and a free spinless particle possessing a momentum $\boldsymbol{k}_0$. Let $\hat{b}^\dagger$ and $\hat{b}$ be the ladder operators for the oscillator. Introduce the creation $\hat{a}^\dagger$ and annihilation $\hat{a}$ operators of Bose type to describe a possible change in the particle's state (note that the further consideration may be applied just as well to a Fermi particle). Input the operator

$$\hat{Q} = \sum_{\boldsymbol{q}} h_{\boldsymbol{q}} \hat{\rho}_{\boldsymbol{q}} \tag{1}$$

where

$$\hat{\rho}_{\boldsymbol{q}} = \sum_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}+\boldsymbol{q}}$$

is the Fourier component of the density operator and $h_{\boldsymbol{q}}$ are coefficients depending on momentum $\boldsymbol{q}$. We can construct another linear combination $\hat{Q}'$ of operators $\hat{\rho}_{\boldsymbol{q}}$ with the help of any other set of coefficients $h'_{\boldsymbol{q}}$. All these combinations are commutative

$$[\hat{Q}, \hat{Q}']_- = 0$$

because the commutation rule

$$[\hat{\rho}_{\boldsymbol{q}}, \hat{\rho}_{\boldsymbol{q}'}]_- = 0 \tag{2}$$

is fulfilled for all $\boldsymbol{q}$ and $\boldsymbol{q}'$.

Input a vector of state $|0, \boldsymbol{k}_0\rangle$, where the first argument (0) denotes a ground state of the oscillator and the second one ($\boldsymbol{k}_0$) describes a state of the particle. Define the vector

$$|h, \boldsymbol{k}_0\rangle = \exp\left(-\tfrac{1}{2}\hat{Q}^\dagger \hat{Q}\right) \sum_{n=0}^{\infty} \frac{1}{n!} (\hat{Q}\hat{b}^\dagger)^n |0, \boldsymbol{k}_0\rangle \tag{3}$$

as an ECS (here we briefly denote by h the whole set of coefficients h_q). Obviously, the vector (3) coincides with the ordinary Schrödinger coherent state (SCS), when one replaces all the particle's operators by their classical equivalents. The ECS describes some state of a system of two interacting quantum objects—the particle and the oscillator. By this circumstance the ECS sufficiently differs from the SCS.

We outline the following general properties of the ECS.

(1) The ECS is not the eigenvector for $\hat{b}$, but

$$\hat{b}|h, \boldsymbol{k}_0\rangle = \hat{Q}|h, \boldsymbol{k}_0\rangle. \tag{4}$$

(2) The operators $\hat{\rho}_q$ only change momenta for all the one-particle states. Hence, the following relations are fullfilled:

$$\hat{\rho}_q|h, \boldsymbol{k}_0\rangle = |h, \boldsymbol{k}_0 - \boldsymbol{q}\rangle \tag{5}$$

$$\hat{\rho}_q^\dagger\hat{\rho}_q|h, \boldsymbol{k}_0\rangle = |h, \boldsymbol{k}_0\rangle. \tag{6}$$

(3) There is the following representation:

$$|h, \boldsymbol{k}_0\rangle = \exp[\hat{Q}\hat{b}^\dagger - \hat{Q}^\dagger\hat{b}]|0, \boldsymbol{k}_0) \tag{7}$$

which is equivalent to the relevant representation of the SCS.

(4) If $h_q = g\Delta(\boldsymbol{q} - \boldsymbol{q}_0)$ we easily have

$$|h, \boldsymbol{k}_0\rangle = \exp\left[\frac{-|g|^2}{2}\right]\sum_{n=0}^{\infty}\frac{g^n}{n!}(\hat{\rho}_{q_0}\hat{b}^\dagger)^n|0, \boldsymbol{k}_0) \tag{8}$$

and, therefore,

$$\langle h, \boldsymbol{k}_0|h', \boldsymbol{k}_0'\rangle = \exp[-\tfrac{1}{2}(|g|^2 + |g'|^2 - 2g^*g')]\Delta(\boldsymbol{k}_0 - \boldsymbol{k}_0'). \tag{9}$$

(5) The total amount of ECS are more than sufficient to define the Hilbert space. Following Klauder [6] (see also [7]) we can introduce the development of the unity operator

$$\hat{I} = \sum_{\boldsymbol{k}}\frac{1}{\pi}\int \mathrm{d}^2z\, \hat{Q}|zh, \boldsymbol{k}\rangle\langle zh, \boldsymbol{k}|\hat{Q}^\dagger \tag{10}$$

where z is the complex variable, $\mathrm{d}^2z = \mathrm{d}[\mathrm{Re}\,(z)]\,\mathrm{d}[\mathrm{Im}\,(z)]$. To prove the last equation one may use the integral

$$\int \mathrm{d}^2z\,(z^*)^n z^m \exp[-|z|^2\hat{Q}^\dagger\hat{Q}]\hat{Q}^{m+1}(\hat{Q}^\dagger)^{n+1} = \pi n!\delta_{nm}.$$

(6) There is the following useful sum rule:

$$\sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}s\boldsymbol{k}}\hat{a}_k|h, \boldsymbol{k}_0\rangle = \mathrm{e}^{\mathrm{i}s\boldsymbol{k}_0}|\alpha) \otimes |\mathrm{vac}_p) \tag{11}$$

where the right-hand side contains a direct product of the SCS for the oscillator

$$|\alpha) = \exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{0}^{\infty}\frac{\alpha^n}{n!}(b^\dagger)^n|0)$$

and a vacuum state of the particle $|\mathrm{vac}_p)$. Here the quantity α is given by the formula

$$\alpha = \sum_{\boldsymbol{q}} h_q\mathrm{e}^{-\mathrm{i}s\boldsymbol{q}}.$$

To prove the property (6) one should keep in mind the relation

$$\sum_{\boldsymbol{k}}\hat{a}_k\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\hat{\rho}_{q_1}\hat{\rho}_{q_2}\ldots|0, \boldsymbol{k}_0) = \mathrm{e}^{\mathrm{i}\boldsymbol{k}_0\boldsymbol{x}}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}_1\boldsymbol{x}}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}_2\boldsymbol{x}}\ldots|0) \otimes |\mathrm{vac}_p) \tag{12}$$

where $|0)$ is the vector of the ground state of the oscillator.

2. Modified perturbation theory

ECSs, first introduced in 1983 [8]†, arise, for example, in a problem of interaction between a moving particle and an oscillator. The proper Hamiltonian can be represented in the following general form:

$$\hat{H}_{\text{int}} = \hat{b}^{\dagger} \sum_{q} g_{q} \hat{\rho}_{q} + \hat{b} \sum_{q} g_{q}^{*} \hat{\rho}_{q}^{\dagger} \tag{13}$$

where g_q is a coupling function. Since $\hat{\rho}_q^{\dagger} = \hat{\rho}_{-q}$, it should be $g_{-q} = g_q^*$.

In most applications the Hamiltonian (13) within the interaction picture depends on time via the density operators $\hat{\rho}(t)$. In these cases we cannot apply ECS without some modification of the theory. Indeed, instead of relations (2) we have

$$[\hat{\rho}_{q}(t), \hat{\rho}_{q'}(t')]_{-} = \sum_{k} \hat{a}_{k}^{\dagger} \hat{a}_{k+q+q'} [\exp\{\mathrm{i}(\varepsilon_{k} t - \varepsilon_{k+q+q'} t' - \mathrm{i}\varepsilon_{k+q}(t - t')\}$$
$$- \exp\{\mathrm{i}(\varepsilon_{k} t' - \varepsilon_{k+q+q'} t + \mathrm{i}\varepsilon_{k+q'}(t - t')\}].$$

We construct a modified perturbation theory with the help of excluding an integrable part of the interaction. For this purpose we expand the operator $\hat{H}_{\text{int}}(t)$ in two parts, $\hat{H}_{\text{int}}^{(0)}(t)$ and $\hat{H}_{\text{int}}^{(1)}(t)$, where

$$\hat{H}_{\text{int}}^{(0)}(t) = \hat{b}^{\dagger} \sum_{q} g_{q} \hat{\rho}_{q} f_{q}(t) + \hat{b} \sum_{q} g_{q}^{*} \hat{\rho}_{q}^{\dagger} f_{q}^{*}(t)$$
$$\hat{H}_{\text{int}}^{(1)}(t) = \hat{H}_{\text{int}}(t) - \hat{H}_{\text{int}}^{(0)}(t).$$

Here the function $f_q(t)$ must be unimodular to preserve the interaction intensity. Obviously, the operators $\sum_q g_q \hat{\rho}_q f_q(t)$, defined at different times, obey the commutation relations. Then, by virtue of the above consideration, the equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|t) = \hat{H}_{\text{int}}^{(0)}(t)|t)$$

acquires an exact solution

$$|t) = \mathrm{e}^{-\mathrm{i}\hat{\chi}(t)}|h, \boldsymbol{k}_0\rangle \tag{14}$$

where $\hat{Q}$ has the previous form (1) and

$$h_{q} = -\mathrm{i} g_{q} \int_{0}^{t} \mathrm{d}t' f_{q}(t') \mathrm{e}^{\mathrm{i}\omega t'}$$
$$\hat{\chi}(t) = -\frac{\mathrm{i}}{2} \int_{0}^{t} \{\hat{\dot{Q}}^{\dagger}(t') \hat{Q}(t') - \hat{Q}^{\dagger}(t') \hat{\dot{Q}}(t')\} \,\mathrm{d}t'.$$

The solution (14) can be rewritten as $|t) = \hat{U}_0(t)|0, \boldsymbol{k}_0\rangle$, where we introduce a zeroth-order evolution operator

$$\hat{U}_{0}(t) = \exp\{\hat{Q}(t)\hat{b}^{\dagger} - \hat{Q}^{\dagger}(t)\hat{b} - \mathrm{i}\hat{\chi}(t)\}.$$

There are the following useful commutation relations:

$$[\hat{b}, \hat{U}_{0}(t)]_{-} = \hat{U}_{0}(t)\hat{Q}(t) \qquad [\hat{b}, \hat{U}_{0}^{\dagger}(t)]_{-} = -\hat{U}_{0}^{\dagger}(t)\hat{Q}(t)$$
$$[\hat{b}^{\dagger}, \hat{U}_{0}(t)]_{-} = \hat{U}_{0}(t)\hat{Q}^{\dagger}(t) \qquad [\hat{b}^{\dagger}, \hat{U}_{0}^{\dagger}(t)]_{-} = -\hat{U}_{0}^{\dagger}(t)\hat{Q}^{\dagger}(t).$$

Let us introduce a new representation for the vector of state and for operators

$$|t\rangle = \hat{U}_{0}^{\dagger}(t)|t) \qquad \tilde{A} = \hat{U}_{0}^{\dagger}(t)\hat{A}\hat{U}_{0}(t).$$

† Extended coherent states were first denoted as 'double coherent' states or 'modified coherent' states.

The new vector of state obeys the equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|t\rangle = \tilde{H}^{(1)}_{\mathrm{int}}(t)|t\rangle$$

which can be solved with the help of a standard technique using the T-exponent

$$|t\rangle = \text{T-exp}\left\{-\mathrm{i}\int_0^t \mathrm{d}t'\,\tilde{H}^{(1)}_{\mathrm{int}}(t')\right\}|0, \boldsymbol{k}_0). \tag{15}$$

If the choice of the function $f_q(t)$ ensures the rapid convergence to the series (15), formula (14) gives a good approximation for the vector of state. In this case we can evaluate a wide set of physical characteristics with sufficient accuracy. As an example, we calculate the density matrix for the particle, for which the exact expression is given by the formula

$$\Gamma(\boldsymbol{x}, \boldsymbol{x}', t) = \langle t|\tilde{\psi}^\dagger(\boldsymbol{x}, t)\tilde{\psi}(\boldsymbol{x}', t)|t\rangle. \tag{16}$$

Here the usual wave operators are introduced, namely,

$$\tilde{\psi}(\boldsymbol{x}, t) = \hat{U}_0^\dagger(t)\hat{\psi}(\boldsymbol{x}, t)\hat{U}_0(t) \qquad \hat{\psi}(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \hat{a}_k \exp\{\mathrm{i}\boldsymbol{k}\boldsymbol{x} - \mathrm{i}\varepsilon_{\boldsymbol{k}} t\}$$

where ε_k is an energy of the particle possessing momentum $\boldsymbol{k}$. The further consideration will be more convenient if the particle–oscillator interaction began at any incident time $t_0 < 0$ when the oscillator was found in the ground state. Let us define the density matrix at $t = 0$. In the first approximation we can set $|t\rangle \approx |0, \boldsymbol{k}_0)$. In this case

$$\Gamma(\boldsymbol{x}, \boldsymbol{x}', t) \approx (0, \boldsymbol{k}_0|\hat{U}_0^\dagger(0)\hat{\psi}^\dagger(\boldsymbol{x}, 0)\hat{\psi}(\boldsymbol{x}', 0)\hat{U}_0(0)|0, \boldsymbol{k}_0). \tag{17}$$

Using relation (7) we have $\hat{U}_0(0)|0, \boldsymbol{k}_0) = \mathrm{e}^{-\mathrm{i}\chi(0)}|h, \boldsymbol{k}_0\rangle$, where $\hat{Q}$ is defined as in (1) with

$$h_q = h_q(0) \qquad h_q(t) = -\mathrm{i}g_q \int_{t_0}^t f_q(t')\mathrm{e}^{\mathrm{i}\omega t'}\,\mathrm{d}t' \qquad t > t_0.$$

Now we apply relations (12) to obtain the formula similar to (11):

$$\hat{\psi}(\boldsymbol{x}', 0)\hat{U}_0(0)|0, \boldsymbol{k}_0) = \exp\{\mathrm{i}\boldsymbol{k}_0\boldsymbol{x}' - \mathrm{i}\Phi(\boldsymbol{x}')\}|\alpha(\boldsymbol{x}', 0)) \otimes |\mathrm{vac}_p) \tag{18}$$

where

$$\alpha(\boldsymbol{x}, t) = \sum_{\boldsymbol{q}} h_q(t)\mathrm{e}^{-\mathrm{i}qx}$$

$$\Phi(\boldsymbol{x}) = \int_{t_0}^0 \mathrm{Im}\,[\dot{\alpha}^*(\boldsymbol{x}, t')\alpha(\boldsymbol{x}, t')]\,\mathrm{d}t'.$$

Substituting (18) into (17) we obtain

$$\Gamma(\boldsymbol{x}, \boldsymbol{x}', 0) \approx \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_0\boldsymbol{x} + \mathrm{i}\boldsymbol{k}_0\boldsymbol{x}'} \exp\{\mathrm{i}\Phi(\boldsymbol{x}) - \mathrm{i}\Phi(\boldsymbol{x}') - \tfrac{1}{2}[|\alpha(\boldsymbol{x}, 0)|^2 + |\alpha(\boldsymbol{x}', 0)|^2 - 2\alpha^*(\boldsymbol{x}, 0)\alpha(\boldsymbol{x}', 0)]\}. \tag{19}$$

Note that in the case $g_q = g\Delta(\boldsymbol{q} - \boldsymbol{q}_0)$, the phase $\Phi(\boldsymbol{x}) = \mathrm{const}$ and formula (19) is simplified.

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